

IN MEMORY OF MY GRANDMOTHER

ON AN INEQUALITY OF DIFFERENT METRICS FOR ALGEBRAIC POLYNOMIALS

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Abstract. We establish an inequality of different metrics for algebraic polynomials.

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1. Introduction and preliminaries

In this section, we give some notation used in the article.

Consider the generalized Jacobi weight

$$\omega_{\alpha,\beta,\gamma}(x) = (1-x)^\alpha(1+x)^\beta|x|^\gamma, \quad x \in [-1, 1],$$

where $\alpha, \beta, \gamma > -1$. Given $1 \leq p \leq \infty$, we denote by $L_p(\omega_{\alpha,\beta,\gamma})$ the space of complex-valued Lebesgue measurable functions f on $[-1, 1]$ with finite norm

$$\|f\|_{L_p(\omega_{\alpha,\beta,\gamma})} = \left(\int_{-1}^1 |f(x)|^p \omega_{\alpha,\beta,\gamma}(x) dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L_\infty(\omega_{\alpha,\beta,\gamma})} = \operatorname{ess\,sup}_{x \in [-1, 1]} |f(x)|.$$

Define the uniform norm of a continuous function f on $[-1, 1]$ by

$$\|f\|_\infty = \max_{-1 \leq x \leq 1} |f(x)|.$$

The maximum and the minimum of two real numbers x and y are denoted by $\max(x, y)$ and $\min(x, y)$, respectively. For $\alpha, \mu \geq 0$, $p \in [1, \infty)$, and $n = 1, 2, \dots$, let

$$l_{\alpha,\mu} = \frac{\alpha}{\alpha + \mu} \quad (l_{0,0} = 0), \quad l_{\alpha,\mu}^{\max} = \frac{\max(\alpha, \mu)}{\alpha + \mu} \quad (l_{0,0}^{\max} = 0),$$

$$C(\alpha, \mu, p, n) = \left(1 - \frac{1}{\pi n}\right)^{-\frac{\min(\alpha, \mu)}{p}} 2^{1+\frac{1}{p}} (\max(\alpha, \mu) + 1)^{\frac{1}{p}} \pi^{\frac{\max(\alpha, \mu)}{p}}. \quad (1)$$

Note that $l_{\alpha,\mu}^{\max} \in [\frac{1}{2}, 1]$ if $\max(\alpha, \mu) > 0$.

The aim of the paper is to establish an inequality of different metrics for algebraic polynomials. In order to realize this aim, we prove a generalization of a lemma by N. K. Bari.

2. Auxiliary results

In this section, we establish some lemmas that will be used to prove our main results.

Lemma 1. *Let $\alpha \geq \mu \geq 0$. Suppose that $\Delta \subset [0, 1]$ is any segment of length l with $l \leq l_{\alpha, \mu}$. Then the following inequality holds:*

$$\int_0^l x^\alpha (1-x)^\mu dx \leq \int_\Delta x^\alpha (1-x)^\mu dx.$$

P r o o f. The claim is obviously true when $\mu = 0$. We now prove the claim for $\mu > 0$.

Note that $l_{\alpha, \mu} \in [\frac{1}{2}, 1)$ when $\mu > 0$. The function $x^\alpha (1-x)^\mu$ is increasing on $[0, \frac{\alpha}{\alpha+\mu}]$ and is decreasing on $[\frac{\alpha}{\alpha+\mu}, 1]$, because the derivative of this function is positive on $(0, \frac{\alpha}{\alpha+\mu})$ and is negative on $(\frac{\alpha}{\alpha+\mu}, 1)$.

Let us consider the following cases:

- I) $\Delta \subset [0, l_{\alpha, \mu}]$;
- II) $\Delta \subset [l_{\alpha, \mu}, 1]$;
- III) $\Delta = [a, b]$ and $l_{\alpha, \mu} \in (a, b)$.

Case I). Because the function $x^\alpha (1-x)^\mu$ is increasing on $[0, \frac{\alpha}{\alpha+\mu}]$, we have, for any segments $\Delta_1 = [a_1, b_1] \subset [0, \frac{\alpha}{\alpha+\mu}]$ and $\Delta_2 = [a_2, b_2] \subset [0, \frac{\alpha}{\alpha+\mu}]$ of equal length with $a_1 \leq a_2$ (or, equivalently, $b_1 \leq b_2$),

$$\int_{\Delta_1} x^\alpha (1-x)^\mu dx \leq \int_{\Delta_2} x^\alpha (1-x)^\mu dx. \quad (2)$$

Putting $\Delta_1 = [0, l]$ and $\Delta_2 = \Delta$ in (2), we obtain the desired inequality.

Case II). Note that in this case $l \leq 1 - l_{\alpha, \mu} \leq \frac{1}{2}$. Because the function $x^\alpha (1-x)^\mu$ is decreasing on $[\frac{\alpha}{\alpha+\mu}, 1]$,

$$\int_\Delta x^\alpha (1-x)^\mu dx \geq \int_{1-l}^1 x^\alpha (1-x)^\mu dx. \quad (3)$$

Since $x^\alpha (1-x)^\mu \leq (1-x)^\alpha x^\mu$ on $[0, \frac{1}{2}]$, we get

$$\int_{1-l}^1 x^\alpha (1-x)^\mu dx = \int_0^l (1-x)^\alpha x^\mu dx \geq \int_0^l x^\alpha (1-x)^\mu dx. \quad (4)$$

Now the desired inequality follows from the inequalities (3) and (4).

Case III). We have

$$\int_0^l x^\alpha (1-x)^\mu dx = \int_0^{b-l_{\alpha, \mu}} x^\alpha (1-x)^\mu dx + \int_{b-l_{\alpha, \mu}}^l x^\alpha (1-x)^\mu dx \quad (5)$$

and

$$\int_{\Delta} x^{\alpha}(1-x)^{\mu} dx = \int_a^{l_{\alpha,\mu}} x^{\alpha}(1-x)^{\mu} dx + \int_{l_{\alpha,\mu}}^b x^{\alpha}(1-x)^{\mu} dx. \quad (6)$$

Applying (2)–(4) in the appropriate settings, we can obtain

$$\begin{aligned} \int_{b-l_{\alpha,\mu}}^l x^{\alpha}(1-x)^{\mu} dx &\leq \int_a^{l_{\alpha,\mu}} x^{\alpha}(1-x)^{\mu} dx, \\ \int_0^{b-l_{\alpha,\mu}} x^{\alpha}(1-x)^{\mu} dx &\leq \int_{l_{\alpha,\mu}}^b x^{\alpha}(1-x)^{\mu} dx. \end{aligned}$$

Using (5), (6), and the above inequalities, we get the desired inequality. \square

Corollary 1. *Let $\mu \geq \alpha \geq 0$. Suppose that $\Delta \subset [0, 1]$ is any segment of length l with $l \leq l_{\mu,\alpha}$. Then the following inequality holds:*

$$\int_{1-l}^1 x^{\alpha}(1-x)^{\mu} dx \leq \int_{\Delta} x^{\alpha}(1-x)^{\mu} dx.$$

Corollary 2. *Let $\alpha, \mu \geq 0$. Suppose that $\Delta \subset [0, 1]$ is any segment of length l with $l \leq l_{\alpha,\mu}^{\max}$ and $l < 1$. Then the following inequality holds:*

$$\int_{\Delta} x^{\alpha}(1-x)^{\mu} dx \geq (1-l)^{\min(\alpha,\mu)} \frac{l^{\max(\alpha,\mu)+1}}{\max(\alpha,\mu)+1}.$$

P r o o f. If $\alpha \geq \mu$, then, by Lemma 1, we get

$$\int_{\Delta} x^{\alpha}(1-x)^{\mu} dx \geq \int_0^l x^{\alpha}(1-x)^{\mu} dx \geq (1-l)^{\mu} \int_0^l x^{\alpha} dx = (1-l)^{\mu} \frac{l^{\alpha+1}}{\alpha+1}. \quad (7)$$

If $\mu \geq \alpha$, then, by Corollary 1, we get

$$\int_{\Delta} x^{\alpha}(1-x)^{\mu} dx \geq \int_{1-l}^1 x^{\alpha}(1-x)^{\mu} dx \geq (1-l)^{\alpha} \int_{1-l}^1 (1-x)^{\mu} dx = (1-l)^{\alpha} \frac{l^{\mu+1}}{\mu+1}. \quad (8)$$

Combining (7) and (8), we obtain the desired estimate. \square

Lemma 2. *Let $\alpha, \mu \geq 0$. Suppose that $\Delta \subset [0, \frac{\pi}{2}]$ is any segment of length l with $l \leq \frac{\pi}{2} l_{\alpha,\mu}^{\max}$ and $l < \frac{\pi}{2}$. Then*

$$\int_{\Delta} |\sin t|^{\alpha} |\cos t|^{\mu} dt \geq \left(1 - \frac{2l}{\pi}\right)^{\min(\alpha,\mu)} \frac{2^{\max(\alpha,\mu)}}{\pi^{\max(\alpha,\mu)}} \cdot \frac{l^{\max(\alpha,\mu)+1}}{\max(\alpha,\mu)+1}.$$

P r o o f. It is well known that, for $t \in [0, \frac{\pi}{2}]$,

$$\sin t \geq \frac{2t}{\pi}, \quad \cos t \geq 1 - \frac{2t}{\pi}.$$

Let $\Delta = [a, b]$. Note that $b - a = l$, $(\frac{2b}{\pi} - \frac{2a}{\pi}) \leq l_{\alpha, \mu}^{\max}$, and $(\frac{2b}{\pi} - \frac{2a}{\pi}) < 1$. Using the above inequalities and Corollary 2, we obtain

$$\begin{aligned} \int_a^b |\sin t|^\alpha |\cos t|^\mu dt &\geq \int_a^b \left(\frac{2}{\pi} t\right)^\alpha \left(1 - \frac{2}{\pi} t\right)^\mu dt = \frac{\pi}{2} \int_{(2a)/\pi}^{(2b)/\pi} x^\alpha (1-x)^\mu dx \geq \\ &\geq \frac{\pi}{2} \left(1 - \frac{2l}{\pi}\right)^{\min(\alpha, \mu)} \cdot \frac{\left(\frac{2}{\pi} l\right)^{\max(\alpha, \mu)+1}}{\max(\alpha, \mu) + 1} = \\ &= \left(1 - \frac{2l}{\pi}\right)^{\min(\alpha, \mu)} \frac{2^{\max(\alpha, \mu)}}{\pi^{\max(\alpha, \mu)}} \cdot \frac{l^{\max(\alpha, \mu)+1}}{\max(\alpha, \mu) + 1}. \end{aligned}$$

□

3. Main results

The following lemma generalizes Lemma 1 in [1].

Lemma 3. *Let $\alpha, \mu \geq 0$, $p \geq 1$, n is a positive integer. For any trigonometric polynomial T_n of degree n , we have*

$$\max_{-\pi \leq t \leq \pi} |T_n(t)| \leq C(\alpha, \mu, p, n) n^{\frac{\max(\alpha, \mu)+1}{p}} \left(\int_{-\pi}^{\pi} |T_n(t)|^p |\sin t|^\alpha |\cos t|^\mu dt \right)^{1/p},$$

where the constant $C(\alpha, \mu, p, n)$ is defined in (1).

P r o o f. Let

$$\nu = |T_n(t_0)| = \max_{-\pi \leq t \leq \pi} |T_n(t)|, \quad (9)$$

$$\Delta_0 = \left[t_0 - \frac{1}{2n}, t_0 + \frac{1}{2n} \right].$$

From Bernstein's inequality it follows that

$$|T'_n(t)| \leq n\nu, \quad t \in [-\pi, \pi]. \quad (10)$$

It is known that, for any $h \geq 0$, there exists a $\theta \in (0, 1)$ such that

$$||T_n(t_0 + h)| - |T_n(t_0)|| \leq |T_n(t_0 + h) - T_n(t_0)| = |h| |T'_n(t_0 + \theta h)|. \quad (11)$$

Using (9)–(11), we get $||T_n(t_0 + h)| - \nu| \leq n\nu|h|$. Hence, for $|h| \leq \frac{1}{2n}$,

$$|T_n(t)| \geq \frac{\nu}{2}, \quad t \in \Delta_0.$$

Thus, we have

$$\begin{aligned} \int_{-\pi}^{\pi} |T_n(t)|^p |\sin t|^\alpha |\cos t|^\mu dt &\geq \int_{\Delta_0} |T_n(t)|^p |\sin t|^\alpha |\cos t|^\mu dt \geq \\ &\geq \left(\frac{\nu}{2}\right)^p \int_{\Delta_0} |\sin t|^\alpha |\cos t|^\mu dt. \end{aligned} \quad (12)$$

Since $|\sin t|^\alpha |\cos t|^\mu$ is an even function of period π , we can assume, without loss of generality, that the centre of Δ_0 belongs to $[0, \frac{\pi}{2}]$. Then there exists a segment Δ of length $\frac{1}{2n}$ such that $\Delta \subset [0, \frac{\pi}{2}]$ and $\Delta \subset \Delta_0$. Note that $\frac{1}{2n} < \frac{\pi}{4} \leq \frac{\pi}{2} l_{\alpha, \mu}^{\max}$. Hence, using Lemma 2, we get

$$\begin{aligned} \int_{\Delta_0} |\sin t|^\alpha |\cos t|^\mu dt &\geq \int_{\Delta} |\sin t|^\alpha |\cos t|^\mu dt \geq \\ &\geq \left(1 - \frac{1}{\pi n}\right)^{\min(\alpha, \mu)} \cdot \frac{1}{2(\max(\alpha, \mu) + 1) \pi^{\max(\alpha, \mu)} n^{\max(\alpha, \mu) + 1}}. \end{aligned} \quad (13)$$

From (12), (13) it follows that

$$\begin{aligned} \left(\int_{-\pi}^{\pi} |T_n(t)|^p |\sin t|^\alpha |\cos t|^\mu dt \right)^{1/p} &\geq \\ &\geq \frac{\nu}{2} \cdot \left\{ \left(1 - \frac{1}{\pi n}\right)^{\min(\alpha, \mu)} \cdot \frac{1}{2(\max(\alpha, \mu) + 1) \pi^{\max(\alpha, \mu)} n^{\max(\alpha, \mu) + 1}} \right\}^{1/p} = \\ &= \left\{ \left(1 - \frac{1}{\pi n}\right)^{-\frac{\min(\alpha, \mu)}{p}} 2^{1+\frac{1}{p}} (\max(\alpha, \mu) + 1)^{\frac{1}{p}} \pi^{\frac{\max(\alpha, \mu)}{p}} n^{\frac{\max(\alpha, \mu) + 1}{p}} \right\}^{-1} \cdot \nu. \end{aligned}$$

□

Now we list some properties of $C(\alpha, \mu, p, n)$:

- (1) $C(\alpha, \mu, p, n) \leq C(\alpha, \mu, p, 1)$, $n = 1, 2, \dots$
- (2) $C(\alpha, \mu, p, n) \rightarrow 2^{1+\frac{1}{p}} (\max(\alpha, \mu) + 1)^{\frac{1}{p}} \pi^{\frac{\max(\alpha, \mu)}{p}}$, $n \rightarrow \infty$.
- (3) If $\max(\alpha, \mu) \leq p$, then $C(\alpha, \mu, p, 1) \leq \frac{8\pi^2}{\pi-1}$ and $C(\alpha, 0, p, 1) \leq 8\pi$.

The following theorem generalizes Lemma (an inequality of different metrics for polynomials) in [3].

Theorem 1. *Let $\alpha \geq \beta \geq -\frac{1}{2}$, $\mu \geq 0$, $1 \leq p < q \leq \infty$, n is a positive integer. If P_n is an algebraic polynomial of degree n , then*

$$\|P_n\|_{L_q(\omega_{\alpha, \beta, \mu})} \leq B(\alpha, \beta, \mu, p, n)^{\left(\frac{1}{p} - \frac{1}{q}\right)} n^{\max(2(\alpha+1), \mu+1)\left(\frac{1}{p} - \frac{1}{q}\right)} \|P_n\|_{L_p(\omega_{\alpha, \beta, \mu})},$$

where

$$B(\alpha, \beta, \mu, p, n) = 2^{2p+1+\alpha-\beta} \left(1 - \frac{1}{\pi n}\right)^{-\min(2\alpha+1, \mu)} \max(2(\alpha+1), \mu+1) \pi^{\max(2\alpha+1, \mu)}.$$

P r o o f. Note that

$$B(\alpha, \beta, \mu, p, n) = \left\{ 2^{1+\frac{\alpha-\beta}{p}} C(2\alpha+1, \mu, p, n) \right\}^p.$$

Using Lemma 3, we get

$$\begin{aligned} \|P_n\|_\infty &= \max_{-\pi \leq t \leq \pi} |P_n(\cos t)| \leq \\ &\leq C(2\alpha+1, \mu, p, n) n^{\frac{\max(2\alpha+1, \mu)+1}{p}} \left(\int_{-\pi}^{\pi} |P_n(\cos t)|^p |\sin t|^{2\alpha+1} |\cos t|^\mu dt \right)^{1/p} = \\ &= 2 C(2\alpha+1, \mu, p, n) n^{\frac{\max(2(\alpha+1), \mu+1)}{p}} \left(\int_0^\pi |P_n(\cos t)|^p (\sin t)^{2\alpha+1} |\cos t|^\mu dt \right)^{1/p} = \\ &= 2^{1+\frac{2\alpha+1}{p}} C(2\alpha+1, \mu, p, n) n^{\frac{\max(2(\alpha+1), \mu+1)}{p}} \times \\ &\quad \times \left(\int_0^\pi |P_n(\cos t)|^p \left(\sin \frac{t}{2} \right)^{2\alpha+1} \left(\cos \frac{t}{2} \right)^{2\alpha+1} |\cos t|^\mu dt \right)^{1/p} = \\ &= 2 C(2\alpha+1, \mu, p, n) n^{\frac{\max(2(\alpha+1), \mu+1)}{p}} \times \\ &\quad \times \left(\int_0^\pi |P_n(\cos t)|^p (1-\cos t)^{\alpha+1/2} (1+\cos t)^{\alpha+1/2} |\cos t|^\mu dt \right)^{1/p} = \\ &= 2^{1+\frac{\alpha-\beta}{p}} C(2\alpha+1, \mu, p, n) n^{\frac{\max(2(\alpha+1), \mu+1)}{p}} \times \\ &\quad \times \left(\int_0^\pi |P_n(\cos t)|^p (1-\cos t)^{\alpha+1/2} (1+\cos t)^{\beta+1/2} |\cos t|^\mu dt \right)^{1/p} = \\ &= 2^{1+\frac{\alpha-\beta}{p}} C(2\alpha+1, \mu, p, n) n^{\frac{\max(2(\alpha+1), \mu+1)}{p}} \times \\ &\quad \times \left(\int_{-1}^1 |P_n(x)|^p (1-x)^\alpha (1+x)^\beta |x|^\mu dx \right)^{1/p} = \\ &= 2^{1+\frac{\alpha-\beta}{p}} C(2\alpha+1, \mu, p, n) n^{\frac{\max(2(\alpha+1), \mu+1)}{p}} \|P_n\|_{L_p(\omega_{\alpha, \beta, \mu})}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|P_n\|_{L_q(\omega_{\alpha, \beta, \mu})} &= \left(\int_{-1}^1 |P_n(x)|^q (1-x)^\alpha (1+x)^\beta |x|^\mu dx \right)^{1/q} \leq \\ &\leq \left(\int_{-1}^1 \|P_n\|_\infty^{q-p} |P_n(x)|^p (1-x)^\alpha (1+x)^\beta |x|^\mu dx \right)^{1/q} = \\ &= \|P_n\|_\infty^{1-\frac{p}{q}} \|P_n\|_{L_p(\omega_{\alpha, \beta, \mu})}^{\frac{p}{q}} \leq \\ &\leq \left\{ \left(2^{1+\frac{\alpha-\beta}{p}} C(2\alpha+1, \mu, p, n) \right)^p \right\}^{\frac{1}{p}-\frac{1}{q}} n^{\max(2(\alpha+1), \mu+1) \left(\frac{1}{p}-\frac{1}{q} \right)} \|P_n\|_{L_p(\omega_{\alpha, \beta, \mu})}. \end{aligned}$$

□

Now we list some properties of $B(\alpha, \beta, \mu, p, n)$:

- (a) $B(\alpha, \beta, \mu, p, n) \leq B(\alpha, \beta, \mu, p, 1)$, $n = 1, 2, \dots$
- (b) $B(\alpha, \beta, \mu, p, n) \rightarrow 2^{2p+1+\alpha-\beta} \max(2(\alpha+1), \mu+1) \pi^{\max(2\alpha+1, \mu)}$, $n \rightarrow \infty$.
- (c) If $\max(2\alpha+1, \mu) \leq p$, then $B(\alpha, \beta, \mu, p, 1) \leq 2^{p+\alpha-\beta} \left(\frac{8\pi^2}{\pi-1} \right)^p$.

4. Conclusion

Our next aim is to prove based on Lemma 2.2 in [2] that the inequality in Theorem 1 is precise in order.

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